

MST125 Solutions to the Specimen Examination Paper

Section A

Detailed solutions are given in this document.

Correct option summary for Section A

The correct options are as follows.

Question	1	2	3	4	5	6	7	8	9
Correct option	D	D	C	A	D	B	B	D	C

Question	10	11	12	13	14	15	16	17	18
Correct option	D	B	D	C	E	C	A	B	A

Section A

Question 1

As 23 is a prime number, and 6 is not a multiple of 23, you can use Fermat's little theorem.

By Fermat's little theorem,

$$6^{22} \equiv 1 \pmod{23}.$$

Since $6^{24} = 6^2 \times 6^{22}$, we obtain

$$\begin{aligned} 6^{24} &\equiv 6^2 \times 6^{22} \\ &\equiv 6^2 \times 1 \\ &\equiv 36 \\ &\equiv 13 \pmod{23}. \end{aligned}$$

References:

Handbook, page 75

Unit 3, Example 9 (Subsection 2.3)

The correct option is D.

Question 2

In order to use the table of properties of a parabola in standard position, you need to rearrange the equation of the parabola into the form $y^2 = 4ax$.

For a parabola in standard position with equation $y^2 = 4ax$, the equation of the directrix is $x = -a$.

Rearranging the equation $4y^2 - 3x = 0$ gives

$$y^2 = \frac{3}{4}x,$$

so here $4a = \frac{3}{4}$; that is, $a = \frac{3}{16}$. Therefore the equation of the directrix is $x = -\frac{3}{16}$.

References:

Handbook, page 80

Unit 4, Example 2 (Subsection 2.3)

The correct option is D.

Question 3

The equation is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with $a^2 = 16$ and $b^2 = 2$. The eccentricity of the ellipse is

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{2}{16}} = \sqrt{1 - \frac{1}{8}} = \sqrt{\frac{7}{8}} = \frac{\sqrt{7}}{2\sqrt{2}}.$$

References:

Handbook, page 80

Unit 4, Activity 9 (Subsection 3.2)

The correct option is C.

Question 4

In each option the parametric equations are of the form

$$x = a + bt, \quad y = c + dt,$$

where a , b , c and d are constants with a and c not both zero, so they describe a line segment whose endpoints correspond to the endpoints of the given range of values of t .

The range of values for t in option A is $0 \leq t \leq 1$. Substituting $t = 0$ into the parametric equations in option A gives

$$x = -1, \quad t = 2;$$

that is, the point $(-1, 2)$. Similarly, substituting $t = 1$ gives

$$x = -1 + 3 \times 1 = 2, \quad t = 2 + 2 \times 1 = 4;$$

that is, the point $(2, 4)$. These are the two points given in the question, so option A is correct.

You can quickly eliminate the other options, too. For example, one of the endpoints of the range of values for t in option B is 2, and substituting $t = 2$ into the equation for y in option B gives $y = 2 + 2 \times 2 = 6$. Since neither of the two points given in the question has y -value 6, option B is incorrect. You can eliminate the other options in a similar way.

You might also notice here that option A is in fact a standard parametrisation. As stated in the Handbook, the line passing through the points (x_0, y_0) and (x_1, y_1) has parametric equations $x = x_0 + t(x_1 - x_0)$, $y = y_0 + t(y_1 - y_0)$. With this parametrisation, $t = 0$ corresponds to the point (x_0, y_0) and $t = 1$ corresponds to the point (x_1, y_1) . Taking $(x_0, y_0) = (-1, 2)$ and $(x_1, y_1) = (2, 4)$ in these parametric equations gives the parametric equations in option A, and the range $0 \leq t \leq 1$ in option A restricts the line to the line segment joining $(-1, 2)$ and $(2, 4)$.

References:

Handbook, page 83

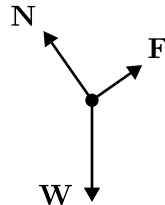
Unit 4, Activity 33(b) (Subsection 6.2)

The correct option is A.

Question 5

The forces acting on the block are

- its weight, $\mathbf{W}$, which acts vertically downwards
- the normal reaction, $\mathbf{N}$, from the surface, which acts perpendicular to the surface
- the friction force, $\mathbf{F}$, from the surface, which acts up the slope (to oppose any possible motion).



References:

Handbook, page 85

Unit 5, Example 10 (Subsection 3.1); see also the first paragraph in Subsection 3.2

The correct option is D.

Question 6

Draw a right-angled triangle with hypotenuse $\mathbf{F}$ and shorter sides parallel to $\mathbf{i}$ and $\mathbf{j}$, work out the lengths of its sides, and hence work out the required components. Alternatively, use the standard formula, as below.

The force $\mathbf{F}$ has magnitude 8.3 N and makes an (anticlockwise) angle of $180^\circ - 25^\circ = 155^\circ$ with the $\mathbf{i}$ -direction, so its component form in newtons is

$$8.3 \cos 155^\circ \mathbf{i} + 8.3 \sin 155^\circ \mathbf{j} = -7.5 \mathbf{i} + 3.5 \mathbf{j} \quad (\text{to 2 s.f.}).$$

References:

Handbook, page 86

Unit 5, Example 7 (Subsection 1.2); see also the paragraph following this example

The correct option is B.

Question 7

Start by finding the images of $(1, 0)$ and $(0, 1)$ under f , and hence find the matrix of f . Deduce facts that enable you to reject some of the options.

We have $f(1, 0) = (0, -1)$ and $f(0, 1) = (-1, 0)$ so f has matrix

$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Since $\det \mathbf{A} = -1$, the transformation f reverses

orientation and therefore cannot be a rotation. So either option A or option B is correct. Only option B gives the correct images for $(1, 0)$ and $(0, 1)$.

References:

Handbook, pages 90 and 92

Unit 6, Example 13 (Subsection 4.1) and Example 18 (Subsection 4.3)

The correct option is B.

Question 8

To find the matrix of $g \circ f$, multiply the matrix of g and the matrix of f , in this order.

The linear transformation $g \circ f$ is represented by the matrix

$$\mathbf{QP} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 5 & -2 \end{pmatrix}.$$

References:

Handbook, page 92

Unit 6, Example 19 (Subsection 5.1)

The correct option is D.

Question 9

You can use the strategy on page 96 of the *Handbook*. The given expression has a linear factor and a repeated linear factor in its denominator.

The linear factor $x - 3$ gives rise to a single partial fraction of the form

$\frac{A}{x - 3}$. The repeated linear factor $(x + 2)^2$ gives rise to two partial fractions

of the form $\frac{B}{x + 2} + \frac{C}{(x + 2)^2}$.

References:

Handbook, page 96

Unit 7, Example 11 (Subsection 2.3)

The correct option is C.

Question 10

The quotient is the polynomial $q(x)$ such that

$$2x^2 + 5x - 3 = q(x)(x + 4) + r(x)$$

where the degree of the remainder $r(x)$ is less than 1. You can work the quotient out using polynomial long division.

The application of polynomial long division is shown below.

$$\begin{array}{r}
 \overline{2x - 3} \\
 x+4 \overline{) 2x^2 + 5x - 3} \\
 \underline{2x^2 + 8x} \\
 -3x - 3 \\
 \underline{-3x - 12} \\
 9
 \end{array}$$

Therefore the quotient is $2x - 3$.

References:

Handbook, page 94

Unit 7, Example 14 (Subsection 3.1)

The correct option is D.

Question 11

The differential equation is of the form

$$\frac{dy}{dx} = f(x)g(y),$$

where $f(x) = \frac{1}{x} + 1$ and $g(y) = y$. So it is a separable differential equation.

Separating the variables in the differential equation gives

$$\int \frac{1}{y} dy = \int \left(\frac{1}{x} + 1 \right) dx.$$

Integrating both sides gives

$$\ln |y| = \ln |x| + x + c,$$

where c is an arbitrary constant. Since $x > 0$ and $y > 0$, we have $|x| = x$ and $|y| = y$, which gives

$$\ln y = \ln x + x + c.$$

Applying the exponential function to both sides of this equation gives

$$\begin{aligned}
 y &= e^{\ln x + x + c} \\
 &= x \times e^x \times e^c \\
 &= A x e^x,
 \end{aligned}$$

where $A = e^c$ is an arbitrary positive constant.

References:

Handbook, page 102

Unit 8, Example 6 (Subsection 2.1)

The correct option is B.

Question 12

The differential equation is of the form

$$\frac{dy}{dx} + g(x)y = h(x)$$

with $g(x) = \frac{1}{x}$. Therefore the integrating factor is

$$p(x) = \exp\left(\int \frac{1}{x} dx\right) = \exp(\ln x) = x.$$

(In this working we obtain $\ln x$ rather than $\ln |x|$ because $x > 0$.)

References:

Handbook, page 102

Unit 8, Example 11 (Subsection 4.3)

The correct option is D.

Question 13

The contrapositive is

if n is not even, then n is not a multiple of 6,

or, equivalently,

if n is odd, then n is not a multiple of 6.

References:

Handbook, page 105

Unit 9, Activity 38 (Subsection 5.2)

The correct option is C.

Question 14

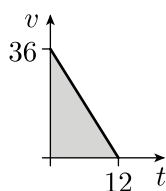
Use an equation of motion for constant acceleration along a straight line.
Measure time from the beginning of the 12 seconds, and choose the positive direction of the x -axis to be the direction of motion, with the origin at the position of the object at time $t = 0$.

We know that the acceleration (in ms^{-2}) is $a = -3$ and the velocity at time $t = 12$ (in seconds) is $v = 0$. We want to find the position x (in metres) of the object at the same time. Using the equation $x = vt - \frac{1}{2}at^2$ gives

$$x = 0 \times 12 - \frac{1}{2} \times (-3) \times 12^2 = \frac{1}{2} \times 3 \times 144 = 216.$$

Hence the distance travelled is 216 m.

(An alternative method is to sketch or imagine the velocity–time graph of the object. It is a straight line with gradient -3 which reaches the time axis ($v = 0$) at time $t = 12$. So at time $t = 0$ the velocity value is $v = 3 \times 12 = 36$.



The change in the position of the object from time $t = 0$ to time $t = 12$ is the area between the graph and the time axis from $t = 0$ to $t = 12$. This is the area of a triangle with base 12 and height 36, so it is $\frac{1}{2} \times 12 \times 36 = 216$.
Hence the distance travelled is 216 m.)

References:

Handbook, page 110 (or page 109 for the alternative method)
Unit 10, Example 3 (Subsection 1.3)

The correct option is E.

Question 15

You need to find the number λ such that

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So you can start by working out the product $\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

We have

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so the required eigenvalue is 2.

References:

Handbook, page 112
Unit 11, Example 1 (Subsection 1.1)

The correct option is C.

Question 16

The sample space is

$$S = \{(1, 1), (1, 2), \dots, (1, 6), \\ (2, 1), (2, 2), \dots, (2, 6), \\ \vdots \\ (6, 1), (6, 2), \dots, (6, 6)\}.$$

It has $|S| = 36$ elements, all equally likely.

The event that two numbers greater than four are rolled is the set

$$E = \{(5, 5), (5, 6), (6, 5), (6, 6)\}.$$

Since $|E| = 4$, we have

$$P(\text{two numbers greater than 4 are rolled on two dice}) \\ = \frac{|E|}{|S|} = \frac{4}{36} = \frac{1}{9}.$$

References:

Handbook, page 118

Unit 12, Example 10 (Subsection 2.4)

The correct option is A.

Question 17

The suggestions for choosing a strategy for solving a counting problem given on page 117 of the Handbook are useful here. In particular, recall that the substitution principle is often useful for problems that contain phrases such as ‘at least one’.

We need to count the elements of the set

S = set of all positive integers less than 500
containing at least one 6.

Let $U = \{1, 2, \dots, 499\}$. Note that $|U| = 499$ and S is a subset of U . The complement of S in U is

$\overline{S}$ = set of all positive integers less than 500 not containing a 6.

To count the set $\overline{S}$, think of every integer in $\overline{S}$ as a 3-digit number, by including leading zeros if necessary. Then there are 5 choices (0, 1, 2, 3 and 4) for the first digit, and 9 choices (any digit except 6) for each of the other two digits, except that this count includes the integer $0 = 000$, which is not in $\overline{S}$. Hence, by the multiplication principle and the subtraction principle,

$$|\overline{S}| = 5 \times 9 \times 9 - 1 = 404.$$

Therefore, by the subtraction principle again,

$$|S| = |U| - |\overline{S}| = 499 - 404 = 95.$$

(You can avoid having to use the subtraction principle twice if you include the number 0 in the universal set U .)

A lengthier alternative solution is to count S directly, using the addition principle, as follows. If you have time, you could try this to check your answer.

Think of every integer in S as a 3-digit number, by including leading zeros if necessary. First consider the integers in S that contain only one 6. Each such integer is of the form $\square 6 \square$ or $\square \square 6$, where the boxes represent digits. In each of these two cases, there are 5 choices (0, 1, 2, 3 and 4) for the digit in the first box and 9 choices (any digit except 6) for the digit in the second box, which gives, by the multiplication principle, $5 \times 9 = 45$ possibilities for each of the two cases. Now consider the integers in S that contain two 6s. Each such integer is of the form $\square 66$, and there are 5 choices (0, 1, 2, 3 and 4) for the digit in the box, giving 5 possibilities. Therefore, by the addition principle,

$$|S| = 45 + 45 + 5 = 95.$$

References:

Handbook, page 117

Unit 12, Example 5 (Subsection 1.3)

The correct option is B.

Question 18

This recurrence system gives x_n in terms of both x_{n-1} and x_{n-2} , so it is a second-order recurrence system. You can use the strategy on page 122 of the Handbook.

The auxiliary equation is

$$r^2 + \frac{3}{2}r - 1 = 0.$$

Rearranging it gives

$$\begin{aligned} 2r^2 + 3r - 2 &= 0 \\ (2r - 1)(r + 2) &= 0 \end{aligned}$$

so the solutions are $r = \frac{1}{2}$ and $r = -2$.

Therefore the general solution is

$$x_n = A \times (-2)^n + B \times \left(\frac{1}{2}\right)^n.$$

References:

Handbook, page 122

Unit 12, Example 16 (Subsection 4.2)

The correct option is A.

Section B

Note on the marking scheme for Sections B and C:

The marking scheme is applied flexibly, to award you a fair mark for your work. For example, in Question 22 you would be awarded the mark for simplifying the numbers as long as you simplify them correctly at some stage in your working – you do not have to simplify them at the particular stage shown in the model solution. If your solution to a question uses a method other than the method in the model solution, but is correct and includes sufficient working, then you would be awarded all the marks for that question, unless the question explicitly asks you to use a method other than the one you used.

Question 19

(a) Euclid's algorithm gives

$$\begin{aligned}87 &= 2 \times 38 + 11 \\38 &= 3 \times 11 + 5 \\11 &= 2 \times 5 + 1 \\5 &= 5 \times 1 + 0.\end{aligned}$$

2 for first three equations

The highest common factor is 1.

1 for answer

(b) Rearranging the equations in part (a) gives

$$\begin{aligned}11 &= 87 - 2 \times 38 \\5 &= 38 - 3 \times 11 \\1 &= 11 - 2 \times 5.\end{aligned}$$

1 for rearranging

Backward substitution gives

$$\begin{aligned}1 &= 11 - 2 \times (38 - 3 \times 11) \\&= 7 \times 11 - 2 \times 38 \\&= 7 \times (87 - 2 \times 38) - 2 \times 38 \\&= 7 \times 87 - 16 \times 38.\end{aligned}$$

3 for working

Therefore $87 \times 7 + 38 \times (-16) = 1$, so $v = 7$ and $w = -16$.

1 for values of v and w

References:

Handbook, pages 73–74

Unit 3, Example 2 (Subsection 1.2) and Example 3 (Subsection 1.3)

Comments:

- In each part, remember to state the final answer as well as carrying out the working.
 - You do not need to write down the rearranged equations before using them in the backwards substitution; you might prefer to do the rearranging in your head. If the backwards substitution is correct, the mark for rearranging will be awarded even if the rearrangement is not explicit.
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Question 20

The suggested substitution is also given on page 100 of the Handbook, where you are given the identity to use once you have made the substitution.

Let $x = 3 \cosh u$. Then $\frac{dx}{du} = 3 \sinh u$. Also $u = \cosh^{-1} \left(\frac{x}{3} \right)$. So

1 for derivative and u in terms of x

$$\int \frac{x^2}{\sqrt{x^2 - 9}} dx$$

$$= \int \frac{9 \cosh^2 u}{\sqrt{9 \cosh^2 u - 9}} (3 \sinh u) du$$

1 for substituting correctly

$$= 9 \int \frac{\cosh^2 u}{\sqrt{\cosh^2 u - 1}} (\sinh u) du$$

1 for simplifying the numbers

$$= 9 \int \frac{\cosh^2 u}{\sqrt{\sinh^2 u}} (\sinh u) du \quad (\text{by the identity } \cosh^2 u - 1 = \sinh^2 u)$$

$$= 9 \int \cosh^2 u du$$

1 for using the identity and simplifying

$$= \frac{9}{2} \int (\cosh(2u) + 1) du \quad (\text{by the half-variable identity for } \cosh)$$

1 for using the half-variable identity

$$= \frac{9}{2} \left(\frac{1}{2} \sinh(2u) + u \right) + c$$

$$= \frac{9}{4} \sinh(2u) + \frac{9}{2} u + c$$

1 for integrating and simplifying

$$= \frac{9}{4} \sinh \left(2 \cosh^{-1} \left(\frac{x}{3} \right) \right) + \frac{9}{2} \cosh^{-1} \left(\frac{x}{3} \right) + c.$$

2 for substituting for u and final answer in x

References:

Handbook, page 100 (and page 6 for hyperbolic identities)
Unit 7, Example 30 (Subsection 6.5)

Comments:

- Remember to include a constant of integration where it is needed.
 - Remember to substitute back for u in terms of x at the end of the working.
 - The expression $\cosh^2 u$ can be integrated by using a half-variable hyperbolic identity; the method is similar to that for integrating $\cos^2 u$ by using a half-angle trigonometric identity.
-

Question 21

(a) When $n = 3$,

$$(n + 1)! = 4! = 24 \quad \text{and} \quad 3^n = 3^3 = 27,$$

so $P(3)$ is false.

1 for check

When $n = 4$,

$$(n + 1)! = 5! = 120 \quad \text{and} \quad 3^n = 3^4 = 81,$$

so $P(4)$ is true.

1 for check

(b) Let $k \geq 4$ and assume that $P(k)$ is true; that is,

$$(k + 1)! > 3^k.$$

$\frac{1}{2}$ for assumption

We want to deduce that $P(k + 1)$ is true; that is,

$$(k + 2)! > 3^{k+1}.$$

$\frac{1}{2}$ for statement to be deduced

Now

$$(k + 2)! = (k + 2) \times (k + 1)!$$

1 for $(k + 2)!$ in terms of $(k + 1)!$

$$> (k + 2) \times 3^k \quad (\text{by } P(k))$$

1 for using $P(k)$

$$> 3 \times 3^k \quad (\text{since } k \geq 4)$$

$$= 3^{k+1}.$$

2 for completing the deduction of $P(k + 1)$

Thus

$$P(k) \Rightarrow P(k + 1), \text{ for } k \geq 4.$$

Since $P(4)$ is true, it follows by mathematical induction that $P(n)$ is true, for $n \geq 4$.

1 for final part of argument

References:

Handbook, page 107

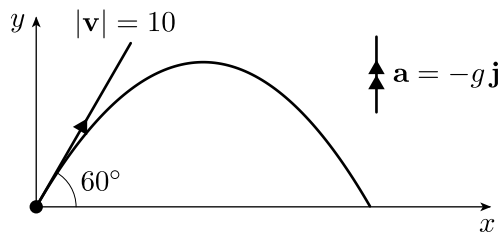
Unit 9, Example 17 (Subsection 4.2)

Comment:

- You must clearly include all parts of the induction argument, such as the final statement.
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Question 22

You might find it helpful to start by making a quick sketch.



- (a) When the ball leaves the ground its velocity is

$$\begin{aligned}\mathbf{v} &= 10 \cos 60^\circ \mathbf{i} + 10 \sin 60^\circ \mathbf{j} \\ &= 10 \times \frac{1}{2} \mathbf{i} + 10 \times \frac{\sqrt{3}}{2} \mathbf{j} \\ &= 5 \mathbf{i} + 5\sqrt{3} \mathbf{j}.\end{aligned}$$

1 for initial velocity

1 for simplifying

- (b) The acceleration of the ball throughout its motion is given by $\mathbf{a} = -g \mathbf{j}$. Hence its velocity is given by

$$\begin{aligned}\mathbf{v} &= \int \mathbf{a} \, dt \\ &= -gt \mathbf{j} + \mathbf{c},\end{aligned}$$

1 for this expression for $\mathbf{v}$

where $\mathbf{c}$ is a vector constant. Since $\mathbf{v} = 5 \mathbf{i} + 5\sqrt{3} \mathbf{j}$ when $t = 0$, this equation gives

1 for using initial conditions

$$5 \mathbf{i} + 5\sqrt{3} \mathbf{j} = -g \times 0 \mathbf{j} + \mathbf{c}.$$

So $\mathbf{c} = 5 \mathbf{i} + 5\sqrt{3} \mathbf{j}$, and hence

$$\mathbf{v} = 5 \mathbf{i} + (5\sqrt{3} - gt) \mathbf{j}.$$

1 for this expression for $\mathbf{v}$

The position of the ball is given by

$$\begin{aligned}\mathbf{r} &= \int \mathbf{v} \, dt \\ &= \int (5 \mathbf{i} + (5\sqrt{3} - gt) \mathbf{j}) \, dt \\ &= 5t \mathbf{i} + \left(5\sqrt{3}t - \frac{1}{2}gt^2 \right) \mathbf{j} + \mathbf{d},\end{aligned}$$

$\frac{1}{2}$ for this integral

1 for this expression for $\mathbf{r}$

where $\mathbf{d}$ is a vector constant. Since $\mathbf{r} = \mathbf{0}$ when $t = 0$, this equation gives $\mathbf{d} = \mathbf{0}$. So the position of the ball is

$\frac{1}{2}$ for using initial conditions

$$\mathbf{r} = 5t \mathbf{i} + \left(5\sqrt{3}t - \frac{1}{2}gt^2 \right) \mathbf{j}.$$

1 for this final expression for $\mathbf{r}$

References:

Handbook, page 111

Unit 10, Example 16 (Subsection 3.2)

Comment:

- In part (a), remember to simplify your answer, since that is possible here.

Question 23

The given matrix is a triangular matrix, so there is a quick way to identify its eigenvalues and eigenvectors.

Since $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ is a triangular matrix, the eigenvalues are the entries on the leading diagonal, namely 2 and 1.

1 for eigenvalues

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 2.

1 for eigenvector corresponding to eigenvalue 2

The eigenvector equations have the form

$$\begin{pmatrix} 2 - \lambda & 3 \\ 0 & 1 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 1$, we obtain

$$\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

1 for this eigenvector equation

This gives

$$x + 3y = 0$$

$$0x + 0y = 0.$$

Hence $x = -3y$, and for $y = 1$ we have $x = -3$. Hence $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 1.

1 for eigenvector corresponding to eigenvalue 1

Let

$$\mathbf{P} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

2 for $\mathbf{P}$ and $\mathbf{D}$

We have $\det \mathbf{P} = 1$, so $\mathbf{P}^{-1} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. Hence

1 for $\mathbf{P}^{-1}$

$$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

1 for this equation

References:

Handbook, pages 112, 113 and 114

Unit 11, Example 9 (Subsection 3.1)

Comments:

- Alternative answers are possible, such as the following:

$$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}.$$

- Remember that the order of the columns in $\mathbf{P}$ must match the order of the eigenvalues in $\mathbf{D}$.
- Don't forget the coefficient $1/\det \mathbf{P}$ when working out the inverse matrix $\mathbf{P}^{-1}$.

Section C

Question 24

(a) The matrix representing f is $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 8 & 3 \end{pmatrix}$.

1 for matrix

(b) The determinant of $\mathbf{A}$ is

$$\det \mathbf{A} = 2 \times 3 - 1 \times 8 = 6 - 8 = -2.$$

Since $\det \mathbf{A} \neq 0$, f is invertible.

1 for showing f invertible

The matrix that represents f^{-1} is

$$\mathbf{A}^{-1} = \frac{1}{-2} \begin{pmatrix} 3 & -1 \\ -8 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ 4 & -1 \end{pmatrix}.$$

1 for matrix of f^{-1}

(c) Each point (x, y) on $f(\mathcal{C})$ is the image under f of the point $f^{-1}(x, y)$, which has position vector

1 for method

$$\mathbf{A}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}x + \frac{1}{2}y \\ 4x - y \end{pmatrix}.$$

1 for $f^{-1}(x, y)$

Hence f maps the point $(-\frac{3}{2}x + \frac{1}{2}y, 4x - y)$ to the point (x, y) . The equation of $f(\mathcal{C})$ is

$$\left(-\frac{3}{2}x + \frac{1}{2}y\right)^2 + (4x - y)^2 = 1.$$

1 for this equation

Multiplying out the brackets gives

$$\frac{9}{4}x^2 - \frac{6}{4}xy + \frac{1}{4}y^2 + 16x^2 - 8xy + y^2 = 1.$$

1 for multiplying out

Multiplying through by 4 gives

$$9x^2 - 6xy + y^2 + 64x^2 - 32xy + 4y^2 = 4,$$

and simplifying gives

$$73x^2 - 38xy + 5y^2 = 4.$$

1 for simplified equation

(d) The point $(1, 3) = f(1, 0)$ lies on $f(\mathcal{C})$, and

1 for point on $f(\mathcal{C})$

$$73 \times 1^2 - 38 \times 1 \times 3 + 5 \times 3^2 = 73 - 114 + 45 = 4,$$

1 for check

as required.

(e) The rule of g is given by

$$g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

$\frac{1}{2}$ for rule of g

Hence the rule of $f \circ g$ is given by

$$\begin{aligned} (f \circ g)(\mathbf{x}) &= f(g(\mathbf{x})) \\ &= \begin{pmatrix} 2 & 1 \\ 8 & 3 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) \\ &= \begin{pmatrix} 2 & 1 \\ 8 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \end{aligned}$$

$\frac{1}{2}$ for this form of the rule

1 for rule in requested form

References:

Handbook, pages 90 and 92

Unit 4, Example 13 (Subsection 4.1), Example 22 (Subsection 5.2) and Example 27 (Subsection 6.1)

Comments:

- In part (b), don't forget the coefficient $1/\det \mathbf{A}$ when working out the inverse matrix $\mathbf{A}^{-1}$.
- When you are simplifying the equation in part (c), it is easier to multiply through to clear the fractions before combining like terms.
- For part (e), remember that both the linear transformation f and the translation g are affine transformations. So you can use the usual method for composing two affine transformations.

Question 25

You can refer to the strategy for sketching the graph of a rational function on page 98 of the Handbook.

- (a) The denominator of $f(x)$ is 0 only when $x^2 - 1 = 0$; that is, when $x = 1$ or $x = -1$. Therefore the domain is

$$(-\infty, -1) \cup (-1, 1) \cup (1, \infty).$$

1 for domain

The y -intercept is $f(0) = 0$.

$\frac{1}{2}$ for y -intercept

The x -intercepts are the solutions of the equation

$$\frac{x}{x^2 - 1} = 0.$$

Therefore the only x -intercept is 0.

$\frac{1}{2}$ for x -intercept

- (b) By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x^2 - 1) \times 1 - x \times 2x}{(x^2 - 1)^2} \\ &= -\frac{x^2 + 1}{(x^2 - 1)^2}. \end{aligned}$$

1 for method

1 for simplified $f'(x)$

- (c) A table of signs for $f'(x)$ is as follows.

	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
$-(x^2 + 1)$	$-$	$-$	$-$	$-$	$-$
$(x^2 - 1)^2$	$+$	0	$+$	0	$+$
$f'(x)$	$-$	$\star$	$-$	$\star$	$-$

$1\frac{1}{2}$ for table of signs

Hence f is decreasing on each of the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.

Since $f'(x)$ is never 0, there are no stationary points.

1 for intervals on which f is decreasing
 $\frac{1}{2}$ for no stationary points

- (d) By part (a), the graph of f has vertical asymptotes $x = -1$ and $x = 1$.

Since $f(x) \rightarrow 0$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, the line $y = 0$ is a horizontal asymptote.

1 for vertical asymptotes

1 for horizontal asymptote

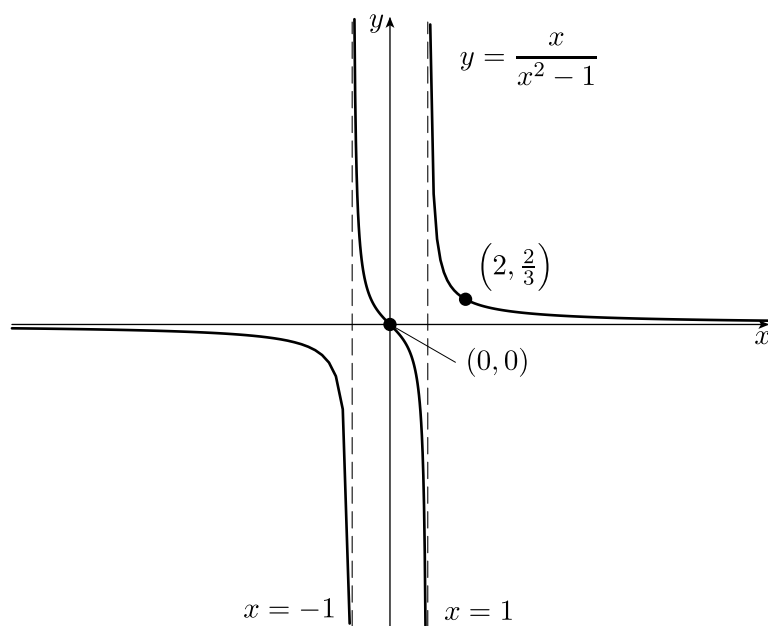
(e) Since

$$f(-x) = \frac{-x}{(-x)^2 - 1} = -\frac{x}{x^2 - 1} = -f(x),$$

f is an odd function.

1 for showing that f is odd

(f)



1 for asymptotes and approach

$\frac{1}{2}$ for intercepts

$\frac{1}{2}$ for shape

References:

Handbook, pages 97–98

Unit 7, Example 22 (Subsection 4.2)

Comment:

- The marked point $(2, \frac{2}{3})$ is included to give an idea of the scale on the y -axis. However, although ideally when you sketch a graph you should include some indication of scale on each axis (often you will do this automatically through the labelling of stationary points and/or asymptotes), in this instance you would not lose marks for not doing so.

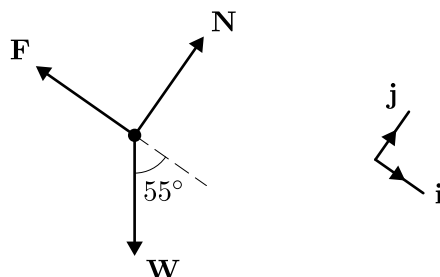
Question 26

You can use the strategy for solving a dynamics problem involving several forces on page 110 of the Handbook.

- (a) The three forces are the friction force $\mathbf{F}$, the normal reaction $\mathbf{N}$ and the weight $\mathbf{W}$ of the box.

$1\frac{1}{2}$ for the three forces

A force diagram is shown below.



- (b) Let m be the mass of the box and let $N = |\mathbf{N}|$. The component forms are

$1\frac{1}{2}$ for force diagram

$$\mathbf{F} = -\mu N \mathbf{i}$$

$$\mathbf{N} = N \mathbf{j}$$

$$\begin{aligned}\mathbf{W} &= mg \cos(-55^\circ) \mathbf{i} + mg \sin(-55^\circ) \mathbf{j} \\ &= mg \cos 55^\circ \mathbf{i} - mg \sin 55^\circ \mathbf{j},\end{aligned}$$

1 for component forms of $\mathbf{F}$ and $\mathbf{N}$

1 for component form of $\mathbf{W}$

where $\mu = 0.17$.

- (c) The resultant force $\mathbf{R}$ acting on the box is

$$\begin{aligned}\mathbf{R} &= -\mu N \mathbf{i} + N \mathbf{j} + mg \cos 55^\circ \mathbf{i} - mg \sin 55^\circ \mathbf{j} \\ &= (mg \cos 55^\circ - \mu N) \mathbf{i} + (N - mg \sin 55^\circ) \mathbf{j}.\end{aligned}$$

1 for resultant force

Since the box moves along the x -axis, its acceleration has component form $a \mathbf{i}$, where a is the $\mathbf{i}$ -component of the acceleration. Applying Newton's second law gives

$$ma \mathbf{i} = (mg \cos 55^\circ - \mu N) \mathbf{i} + (N - mg \sin 55^\circ) \mathbf{j}.$$

1 for applying Newton's second law

Resolving gives

$$ma = mg \cos 55^\circ - \mu N$$

$$0 = N - mg \sin 55^\circ.$$

Hence

$$N = mg \sin 55^\circ$$

1 for expression for N

and therefore

$$ma = mg \cos 55^\circ - \mu mg \sin 55^\circ,$$

1 for expression for ma

which gives

$$\begin{aligned}a &= g \cos 55^\circ - \mu g \sin 55^\circ \\ &= 9.8 \cos 55^\circ - 0.17 \times 9.8 \sin 55^\circ \\ &= 4.256 \dots\end{aligned}$$

The magnitude of the acceleration is 4.3 m s^{-2} (to 2 s.f.).

1 for magnitude of acceleration

(d) Using the equation $x = v_0t + \frac{1}{2}at^2$ with $v_0 = 0$, $x = 20$ and $a = 4.256 \dots$ gives

$$20 = 0 + \frac{1}{2} \times 4.256 \dots \times t^2,$$

1 for method

so

$$t = \sqrt{\frac{2 \times 20}{4.256 \dots}} = 3.065 \dots$$

The box takes 3.1 seconds (to 2 s.f.) to reach the bottom of the slope.

1 for time

References:

Handbook, page 110

Unit 10, Example 11 (Subsection 2.3)

Comments:

- Remember to define any symbols that you introduce, such as N .
 - Don't forget to include units in your final answers to parts (c) and (d).
 - If a problem involves constant acceleration along a straight line, as does the problem in part (d), use the equations of motion for this type of motion.
-